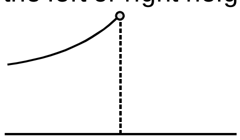
Hence the limit value of f(x) from left of x = 1 should either be greater than or equal to the value of function at x = 1.

$$\lim_{x \to 1^-} f(x) \geq f(1)$$

$$\Rightarrow \quad -1 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} > -1 \qquad \Rightarrow \quad \frac{(b^2+1)(b-1)}{(b+1)\,(b+2)} \geq 0$$

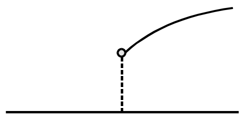
$$\Rightarrow \quad b \in (-2, 1) \cup [1, -\infty)$$

Note : If x = a happens to be a boundary point of the function, then compare the value of f(a) with appropriate values in either the left or right neighbourhood of x = a.



x=a

Local Maxima



x=a

Local Minima

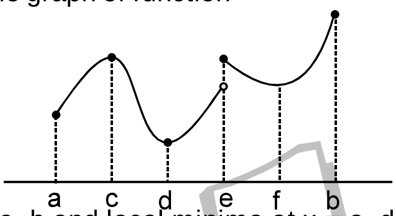
From these figure we can see that boundary points are almost always points of local maxima/ minima.

B. Global Maxima/Minima

Global maximum or minimum value of f(x), x ∈ [a, b] basically refers to the greatest value and least value of f(x) over that interval mathematically

(i) If f(c) ≥ f(x) for ∀ x ∈ [a, b] then f(c) is called global maximum or absolute maximum value of f(x).

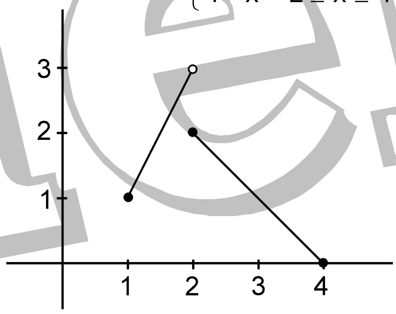
(ii) Similarly if f(d) ≤ f(x) ∀ x ∈ [a, b] then f(d) is called global minimum or absolute minimum value. For example consider the graph of function



f(x) has local maxima at x = c, e, b and local minima at x = a, d, f. It can also be easily seen that f(b) is the greatest value and hence global maximum and similarly f(d) is global minimum.

Also be careful about the fact that a function has global maximum or minimum value when it actually achieves these values.

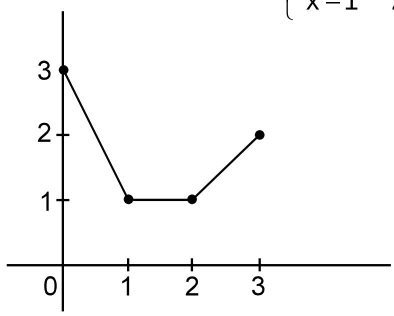
Let us take graph of function as $f(x) = \begin{cases} 2x - 1 & 1 \leq x < 2 \\ 4 - x & 2 \leq x \leq 4 \end{cases}$



This function has local minima at x = 1, 4 and at x = 2, it is a monotonically decreasing function and hence neither maximum nor minimum.

f(4) = 0, which the global minimum value but global maximum value is not defined. The value of function can be made as close to 3 as we may please.

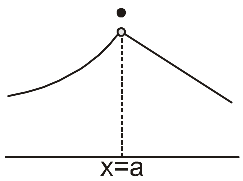
Also consider graph of another function as shown $f(x) \begin{cases} 3 - 2x & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \\ x - 1 & 2 \leq x \leq 3 \end{cases}$

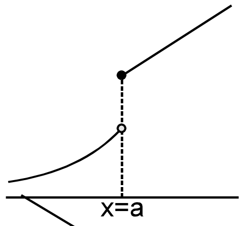


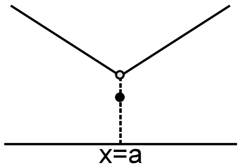
f(x) has local maxima at x = 0, 3 and f(0) = 3 value 1 over this interval which is global minimum although note that f(x) does not has local minima at x = 1, 2.

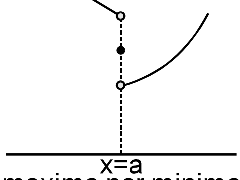
Self Practice Problems

1. In each of following case identify if x = a is point of local maxima, minima or neither of them

(i) 

(ii) 

(iii) 

(iv) 

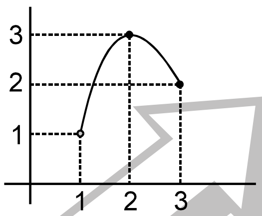
Ans. (i) Maxima (ii) Neither maxima nor minima
(iii) Minima (iv) Neither maxima nor minima

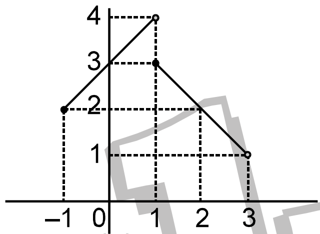
2. If f(x) = $\begin{cases}(x+\lambda)^2 & x<0\\ \cos x & x\geq 0\end{cases}$, find possible values of λ such that f(x) has local maxima at x = 0. **Ans.** $\lambda \in [-1, 1)$

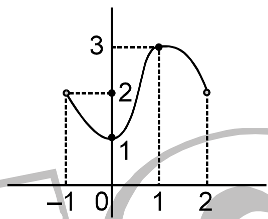
3. Draw the graph of function f(x) = 2 |x – 2| + 5 |x – 3| ($x \in R$). Also identify points of local Maxima/Minima and also global Maximum/Minimum values

Ans. Local minima at x = 3, Global minimum value 2 at x = 3, No point of local maximum, Global maximum value is not defined.

4. Examine the graph of following functions in each case identify the points of global maximum/minimum and local maximum / minimum.

(i) 

(ii) 

(iii) 

Ans. (i) Local maxima at x = 2, Local minima at x = 3, Global maxima at x = 2
(ii) Local minima at x = – 1, No point of Global minima, no point of local or Global maxima
(iii) Local & Global maxima at x = 1, Local & Global minima at x = 0.

C. IInd Fundamental Theorem

Following points should be examined for maxima/minima in an interval.

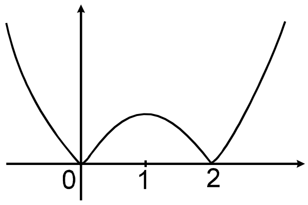
1. Points where $f'(x) = 0$
2. Points where $f'(x)$ does not exists
3. Boundary points of interval (only when the interval is closed).

Example : Find the possible points of Maxima/Minima for f(x) = |x² – 2x| ($x \in R$)

Solution. $f(x) = \begin{cases} x^2-2x & x\geq 2\\ 2x-x^2 & x<x<2 \\ x^2-2x & x\leq 0\end{cases}$

$f'(x) = \begin{cases} 2(x-1) & x>2\\ 2(1-x^2) & 0<x<2 \\ 2(x-1) & x<0\end{cases}$

$f'(x) = 0$ at x = 1 and $f'(x)$ does not exist at x = 0, 2. Thus these are the possible critical points.

f(x) = |x² – 2x| 

from graph we can see that x = 1 is a point of local mixima where as x = 0, 2 are points of local minima.

Example : If f(x) = x³ + ax² + bx + c has extreme values at x = – 1 and x = 3. Find a, b, c.

Solution. Extreme values basically mean maximum or minimum values, since f(x) is differentiable function so

$f'(-1) = 0 = f'(3)$
$f'(x) = 3x^2 + 2ax + b$
$f'(3) = 27 + 6a + b = 0$
$f'(-1) = 3 - 2a + b = 0$ $\Rightarrow$ $a = -3,\ b = -9,\ c \in R$

D Critical Points

All those points in the interior of an interval where $f'(x)$ is either equal to zero or does not exist are called critical points.

Example: Find the critical points of the function f(x) = 4x³ – 6x² – 24x + 9 if (i) $x \in [0, 3]$ (ii) $x \in [-3, 3]$ (iii) $x \in [-1, 2]$.

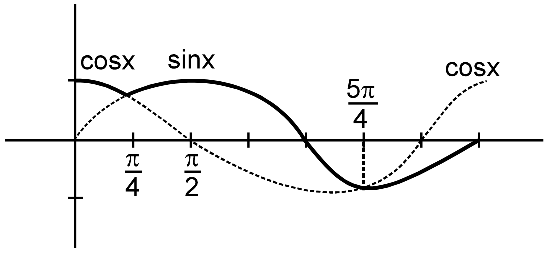
Solution. $f'(x) = 12(x^2 - x - 2)$
$= 12(x - 2)(x + 1)$
$f'(x) = 0 \Rightarrow x = -1$ or 2

(i) if $x \in [0, 3]$, $x = 2$ is the critical point.
(ii) if $x \in [-3, 3]$, then we have two critical points $x = -1, 2$.
(iii) If $x \in [-1, 2]$, then no critical point as both $x = 1$ and $x = 2$ become boundary points.

Note : Critical points are always interior points of an interval.

Example : Find the number of critical points for $f(x) = \max(\sin x, \cos x)$ f, $x \in (0, 2\pi)$.

Solution.



f(x) has three critical points $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$.

D. Test for Maxima/Minima

Upto now we have been able to identity exactly which points should be examined for finding the extreme values of a function. Let as now consider the various tests by which we can separate the critical points into points of local maxima or minima.

1. I[st] derivative Test

(i) If $f'(x)$ changes sign from negative to positive while passing through $x = a$ from left to right then $x = a$ is a point of local maxima
(ii) If $f'(x)$ changes sign from positive to negative while passing through $x = a$ from left to right then $x = a$ is a point of local minima.
(iii) If $f'(x)$ does not changes its sign about $x = a$ then $x = a$ is neither a point of maxima nor minima.

Note : This test is applicable only for continuous functions. If f(x) is discontinuous at x = a, then use of I[st] fundamental theorem is advisable for investigating maxima/minima.

Example : Find the points of maxima or minima of $f(x) = x^2 (x-2)^2$.

Solution. $f(x) = x^2 (x-2)^2$
$f'(x) = 4x (x-1) (x-2)$
$f'(x) = 0 \Rightarrow x = 0, 1, 2$
examining the sign change of $f'(x)$

$-$ 0 $+$ 1 $-$ 2 $+$
0: Minima, 1: Mixima, 2: Minima

Hence x = 1 is point of maxima, x = 0, 2 are points of minima.

Note : In case of continuous functions points of maxima and minima are alternate.

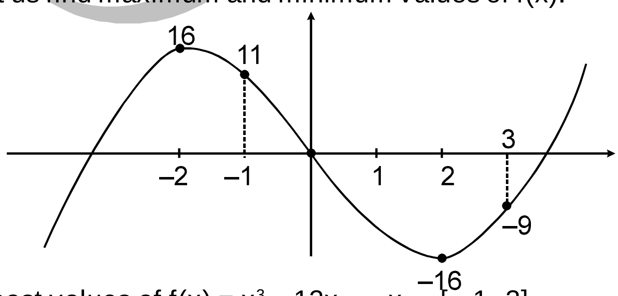
Example : Find the points of Maxima/Minima of $f(x) = x^3 - 12x$ also draw the graph of this functions.

Solution. $f(x) = x^3 - 12x$
$f'(x) = 3(x^2 - 4) = 3(x-2)(x+2)$
$f'(x) = 0 \Rightarrow x = \pm 2$

$+$ -2 $-$ 2 $+$
-2: Maxima, 2: Minima

For tracing the graph let us find maximum and minimum values of f(x).

x	f(x)
2	−16
−2	+16



Example : Find the greatest and least values of $f(x) = x^3 - 12x$ $x \in [-1, 3]$

Solution. By graph of the function $f(x) = x^3 - 12x$ we can easily see that minimum value of f(x) is – 16 and maximum value is 11.

Aliter

We can use II[nd] fundamental theorem. The possible points of maxima/minima are critical points and the boundary points.

for $x \in [-1, 3]$ and $f(x) = x^3 - 12x$
$x = 2$ is the only critical points.

Hence points of local maxima/minima are x = – 1, 2, 3. Examining the value of f(x) at these points we can find greatest and least values.

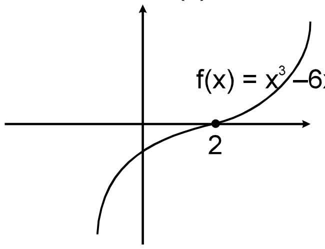
x	f(x)
−1	11
2	−16
3	−9

$\therefore$ Minima f(x) = – 16 & Maxima f(x) = 11.

Example: Show that $f(x) = (x^3 - 6x^2 + 12x - 8)$ does not have any point of local maxima or minima.

Solution. $f(x) = x^3 - 6x^2 + 12x - 8$
$f'(x) = 3(x^2 - 4x + 4)$
$f'(x) = 3(x-2)^2$
$f'(x) = 0 \Rightarrow x = 2$

but clearly $f'(x)$ does not change sign about $x = 2$. $f'(2^+) > 0$ and $f'(2^-) > 0$. So $f(x)$ has no point of maxima or minima. In fact $f(x)$ is a monotonically increasing function for $x \in R$.



Example : Let $f(x) = \begin{cases} x^3 + x^2 - 10x & x < 0 \\ 3\sin x & x \geq 0 \end{cases}$. Examine the behaviour of $f(x)$ at $x = 0$.

Solution. $f(x)$ is continuous at $x = 0$.

$$f'(x) = \begin{cases} 3x^2 + 2x - 10 & x < 0 \\ 3\cos x & x > 0 \end{cases}$$

$f'(0^+) = 3$ and $f'(0^-) = -10$ thus $f(x)$ is non-diff. at $x = 0$ $\Rightarrow$ $x = 0$ is a critical point.
Also derivative changes sign from negative to positive. So $x = 0$ is a point of local minima.

Example : Let $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If $f(x)$ has positive point of maxima, then find possible value of 'a'.

Solution. $f'(x) = 3[x^2 + 2(a - 7)x + (a^2 - 9)] = 0$
Let α, β be roots of $f'(x) = 0$ and let α be the smaller root. Examining sign change of $f'(x)$.

$$\overset{+}{\longrightarrow}\ \alpha\ \overset{-}{\longrightarrow}\ \beta\ \overset{+}{\longrightarrow}$$

Maxima occurs at smaller root α which has to be positive. This basically implies that both of roots $f'(x) = 0$ must be positive.
Applying location of roots

(i) $D > 0$ $\Rightarrow$ $a < \frac{29}{7}$

(ii) $-\frac{b}{2a} > 0$ $\Rightarrow$ $a < 7$

(iii) $f'(0) > 0$ $\Rightarrow$ $a \in (-\infty, -3) \cup (3, \infty)$

from (i), (ii) and (iii) $\Rightarrow$ $a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

Self Practice Problems :

1. Let $f(x) = 2x^3 - 9x^2 + 12x + 6$
(i) Find the possible points of Maxima/Minima of $f(x)$ for $x \in R$.
(ii) Find the number of critical points of $f(x)$ for $x \in [0, 2]$.
(iii) Discuss absoluble Maxima/Minima value of $f(x)$ for $x \in [0, 2]$
(iv) Prove that for $x \in (1, 3)$, the function does not has a Global maximum.
Ans. (i) $x = 1, 2$
(ii) 1 $(x = 1)$
(iii) $f(0) = 6$ is the global minimum, $f(1) = 11$ is global maximum

2. Let $f(x) = \sin x\,(1 + \cos x)$; $x \in (0, 2\pi)$. Find the number of critical points of $f(x)$. Also identify which of these critical points are points of Maxima/Minima.
Ans. 3 critical point $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$
Local maxima at $x = \frac{\pi}{3}$, Local minima at $x = \frac{5\pi}{3}$.

3. Let $f(x) = \frac{x}{2} + \frac{2}{x}$. Find local maximum and local minimum value of $f(x)$. Can you explain this discrepancy of locally minimum value being greater than locally maximum value.
Ans. Local maxima at $x = -2$ $f(-2) = -2$
Local minima at $x = 2$ $f(2) = 2$.

4. Find the points of local Maxima or Minima of following functions
(i) $f(x) = (x - 1)^3 (x + 2)^2$ (ii) $f(x) = \sin 2x - x$
(iii) $f(x) = x^3 + x^2 + x + 1$.
Ans. (i) Maxima at $x = -2$, Minima at $x = 0$
(ii) Maxima at $x = n\pi + \frac{\pi}{6}$; Minima at $x = n\pi - \frac{\pi}{6}$
(iii) No point of local maxima or minima.

2. IInd derivative Test
If $f(x)$ is continuous function in the neighbourhood of $x = 0$ such that $f'(x) = 0$ and $f''(a)$ exists then we can predict maxima or minima at $x = 0$ by examining the sign of $f''(a)$
(i) If $f''(a) > 0$ then $x = a$ is a point of local minima.
(ii) If $f''(a) < 0$ then $x = a$ is a point of local maxima.
(iii) If $f''(a) = 0$ then second derivative test does not gives use conclusive results.

Example : Find the points of local maxima or minima for $f(x) = \sin 2x - x$, $x \in (0, \pi)$.

Solution. $f(x) = \sin 2x - x$
$f'(x) = 2\cos 2x - 1$
$f'(x) = 0$ $\Rightarrow$ $\cos 2x = \frac{1}{2}$ $\Rightarrow$ $x = \frac{\pi}{6}, \frac{5\pi}{6}$

$f''(x) = -4 \sin 2x$

$f''\left(\frac{\pi}{6}\right) < 0 \quad \Rightarrow \quad$ Maxima at $x = \frac{\pi}{6}$

$f''\left(\frac{5\pi}{6}\right) > 0 \quad \Rightarrow \quad$ Minima at $x = \frac{5\pi}{6}$

3. **n^{th} derivative test**

Let f(x) be function such that $f'(a) = f''(a) = f'''(a) = \ldots\ldots = f^{n-1}(a) = 0$ & $f^n(a) = 0$, then

n = even,
(i) $f^n(a) > 0 \Rightarrow$ Minima
(ii) $f^n(a) < 0 \Rightarrow$ Maxima

n = odd
Neither Maxima nor Minima at x = a

Example : Find points of local maxima or minima of $f(x) = x^5 - 5x^2 + 5x^3 - 1$

Solution.
$f(x) = x^5 - 5x^2 + 5x^3 - 1$
$f'(x) = 5x^2 (x - 1)(x - 3)$
$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 3$
$f''(x) = 10x (2x^2 - 6x + 3)$
Now, $f''(1) < 0 \quad \Rightarrow \quad$ Maxima at x = 1
$f''(3) > 0 \quad \Rightarrow \quad$ Minima at x = 3
and, $f''(0) = 0 \quad \Rightarrow \quad$ II^{nd} derivative test fails
so, $f'''(x) = 30 (2x^2 - 4x + 1)$
$f'''(0) = 30 \quad \Rightarrow \quad$ Neither maxima nor minima at x = 0.

Note : It was very convenient to check maxima/minima at first step by examining the sign change of $f'(x)$ no sign change of $f'(x)$ at x = 0

$f'(x) = 5x^2 (x - 1)(x - 3)$

+ + – +
0 1 3
max. min.

E. Application of Maxima/Minima to Problems

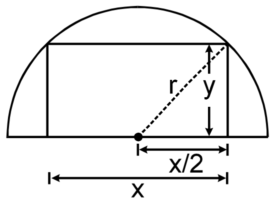
Example : Find two positive numbers x and y such that x + y = 60 and xy^3 is maximum.

Solution.
$x + y = 60$
$\Rightarrow \quad x = 60 - y \quad \Rightarrow \quad xy^3 = (60 - y)y^3$
Let $f(y) = (60 - y)\,y^3 \quad ; \quad y \in (0, 60)$
for maximizing f(y) let us find critical points
$f'(y) = 3y^2 (60 - y) - y^3 = 0$
$f'(y) = y^2 (180 - 4y) = 0$
$\Rightarrow \quad y = 45$
$f'(45^+) < 0$ and $f'(45^-) > 0$. Hence local maxima at y = 45.
So $x = 15$ and $y = 45$.

Example : Rectangles are inscribe inside a semi-circle of radius r. Find the rectangle with maximum area.

Solution. Let sides of rectangle be x and y.
$\Rightarrow \quad A = xy.$
Here x and y are not independent variables and are related by pythogoreas theorem with r.

$$\frac{x^2}{4} + y^2 = r^2 \quad \Rightarrow \quad y = \sqrt{r^2 - \frac{x^2}{4}}$$



$$\Rightarrow \quad A(x) = x\sqrt{r^2 - \frac{x^2}{4}}$$

$$\Rightarrow \quad A(x) = \sqrt{x^2 r^2 - \frac{x^4}{4}}$$

Let $f(x) = r^2x^2 - \frac{x^4}{4}$; $\quad x \in (0, r)$

A(x) is maximum when f(x) is maximum

Hence $f'(x) = x(2r^2 - x^2) = 0$

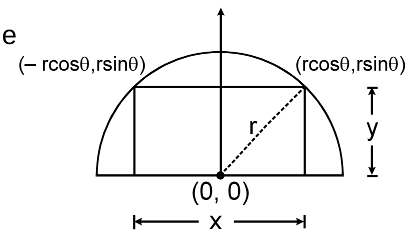
$\Rightarrow \quad x = r\sqrt{2}$

also $f'(r\sqrt{2^+}) < 0$ and $f'(r\sqrt{2^-}) > 0$

confirming at f(x) is maximum when $x = r\sqrt{2}$ & $y = \frac{r}{\sqrt{2}}$.

Aliter Let use choose coordinate system with origin as centre of circle

$A = xy$



$\Rightarrow \quad A = 2\,(r\cos\theta)\,(r\sin\theta)$

$\Rightarrow \quad A = r^2 \sin 2\theta \qquad \theta \in \left(0, \frac{\pi}{2}\right)$

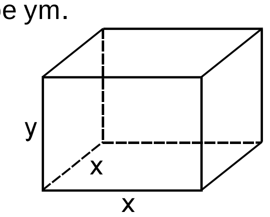
Clearly A is maximum when $\theta = \frac{\pi}{4}$

$\Rightarrow \quad x = r\sqrt{2} \quad$ and $\quad y = \frac{r}{\sqrt{2}}$.

Example: A sheet of area 40 m^2 in used to make an open tank with square base. Find the dimensions of the base such that volume of this tank is maximum.

Solution. Let length of base be xm and height be ym.

$v = x^2y$



again x and y are related to surface area of this tank which is equal to 40 m^2.

$\Rightarrow \quad x^2 + 4xy = 40$

$$y = \frac{40 - x^2}{4x} \qquad x \in (0, \sqrt{40}\,) \quad \Rightarrow \quad V(x) = x^2\left(\frac{40 - x^2}{4x}\right)$$

$$V(x) = \frac{(40x - x^3)}{4}$$

maximizing volume,

$$V'(x) = \frac{(40 - 3x^2)}{4} = 0 \qquad \Rightarrow \qquad x = \sqrt{\frac{40}{3}}\ \text{m}$$

and $\quad V''(x) = -\frac{3x}{2} \qquad \Rightarrow \qquad V''\left(\frac{\sqrt{40}}{3}\right) < 0.$

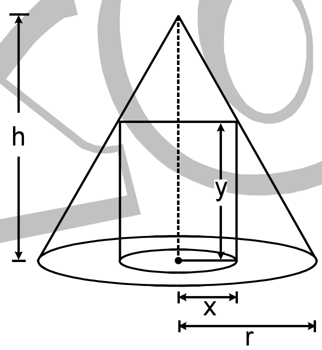
confirming that volume is maximum at $x = \frac{\sqrt{40}}{3}$ m.

Example : If a right circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

Solution. Let x be the radius of cylinder and y be its height

$v = \pi x^2 y$

x, y can be related by sing similar triangles



$$\frac{y}{r - x} = \frac{h}{r}$$

$\Rightarrow \quad y = \frac{h}{r}\,(r - x)$

$\Rightarrow \quad v(x) = \pi x^2\,\frac{h}{r}\,(r - x) \qquad x \in (0, r)$

$\Rightarrow \quad v(x) = \frac{\pi h}{r}\,(rx^2 - x^3)$

$$v'(x) = \frac{\pi h}{r}\,x\,(2r - 3x)$$

$v'(x) = 0 \qquad$ and $\qquad v'\left(\frac{2r}{3}\right) > 0$

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

Note : Following formulae of volume, surface area of important solids are very useful in problems of maxima & minima.

6. Useful Formulae of Measuration to Remember :

1. Volume of a cuboid = ℓbh.

2. Surface area of cuboid = 2(ℓb + bh + hℓ).

3. Volume of cube = a^3

4. Surface area of cube = $6a^2$

5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.

6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height)

7. Curved surface of a cylinder = $2\pi rh$.

8. Total surface of a cylinder = $2\pi rh + 2\pi r^2$.

9. Volume of a sphere = $\frac{4}{3}\pi r^3$.

10. Surface area of a sphere = $4\pi r^2$.

11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

12. Volume of a prism = (area of the base) × (height).

13. Lateral surface of a prism = (perimeter of the base) × (height).

14. Total surface of a prism = (lateral surface) + 2 (area of the base)
(Note that lateral surfaces of a prism are all rectangle).

15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).

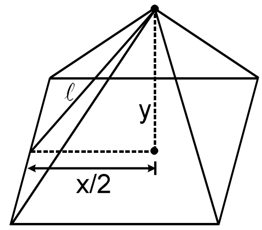
16. Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
(Note that slant surfaces of a pyramid are triangles).

Example : Among all regular square pyramids of volume $36\sqrt{2}$ cm³. Find dimensions of the pyramid having least lateral surface area.

Solution. Let the length of a side of base be x cm and y be the perpendicular height of the pyramid

$$V = \frac{1}{3} \text{ area of base x height}$$

$$\Rightarrow \quad V = \frac{1}{3}x^2y = 36\sqrt{2}$$



$$\Rightarrow \quad y = \frac{108\sqrt{2}}{x^2}$$

and $\quad S = \frac{1}{2}$ perimeter of base x slant height

$$= \frac{1}{2}(Ax).\ \ell$$

but $\quad \ell = \sqrt{\frac{x^2}{4} + y^2}$

$$\Rightarrow \quad S = 2x\sqrt{\frac{x^2}{4}+y^2} = \sqrt{x^4+4x^2y^2} \qquad \Rightarrow \qquad S = \sqrt{x^4+4x^2\left(\frac{108\sqrt{2}}{x^2}\right)^2}$$

$$S(x) = \sqrt{x^4 + \frac{8.(108)^2}{x^2}}$$

Let $\quad f(x) = x^4 + \frac{8.(108)^2}{x^2}$ for minimizing f(x)

$$f'(x) = 4x^3 - \frac{16(108)^2}{x^3} = 0$$

$$\Rightarrow \quad f'(x) = 4\ \frac{(x^6 - 6^6)}{x^3} = 0$$

$\Rightarrow \quad$ x = 6, which a point of minima

Hence x = 6 cm and y = $3\sqrt{2}$.

Example : Let A(1, 2) and B(– 2, – 4) be two fixed points. A variable point P is chosen on the straight line y = x such that perimeter of ΔPAB is minimum. Find coordinates of P.

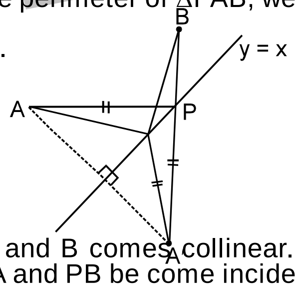
Solution. Since distance AB is fixed so for minimizing the perimeter of ΔPAB, we basically have to minimize (PA + PB)

Let A′ be the mirror image of A in the line y = x.

F(P) = PA + PB

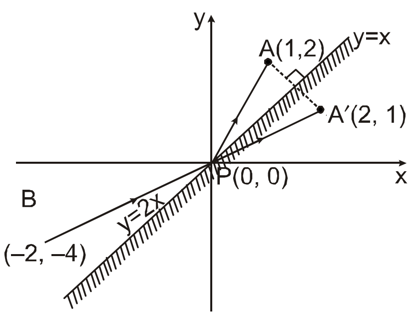
F(P) = PA′ + PB

But for ΔPA′B



PA′ + PB ≥ A′B and equality hold when P, A′ and B comes collinear. Thus for minimum path length point P is that special point for which PA and PB be come incident and reflected rays with respect to the mirror y = x.

Equation of line joining A′ and B is y = 2x intersection of this line with y = x is the point P. Hence P ≡ (0, 0).



Note : Above concept is very useful because such problems become very lengthily by making perimeter as a function of position of P and then minimizing it.

Self Practice Problems :

1. Find the two positive numbers x and y whose sum is 35 and the product $x^2 y^5$ maximum.
Ans. x = 25, y = 10.

2. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the slops to form a box. What should be the side of the square to be cut off such that volume of the box is maximum possible. **Ans.** 3 cm

3. Prove that a fight circular cylinder of given surface area and maximum volume is such that the height is equal to the diameter of the base.

4. A normal is drawn to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$. Find the maximum distance of this normal from the centre.
Ans. 1 unit

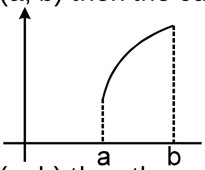
5. A line is drawn passing through point P(1, 2) to cut positive coordinates axes at A and B. Find minimum area of ΔPAB. **Ans.** 4 units

6. Two towns A and B are situated on the same side of a straight road at distances a and b respectively perpendiculars drawn from A and B meet the road at point c and d respectively. The distance between C and D is C. A hospital is to be built at a point P on the road such that the distance APB is minimum. Find position of P. **Ans.** P is at distance of $\frac{ac}{a+b}$ from c.

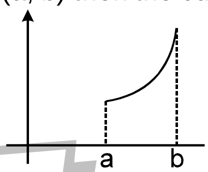
F. Points of Inflection

For continuous function f(x), If $f''(x_0) = 0$ or doesnot exist at points where $f'(x_0)$ exists and if $f''(x)$ changes sign when passing through $x = x_0$ then x_0 is called a point of inflection. At the point of inflection, the curve changes its concavity i.e.

(i) If $f''(x) < 0$, $x \in (a, b)$ then the curve y = f (x) is convex in (a, b)



(ii) If $f''(x) > 0$, $x \in (a, b)$ then the curve y = f (x) is concave in (a, b)

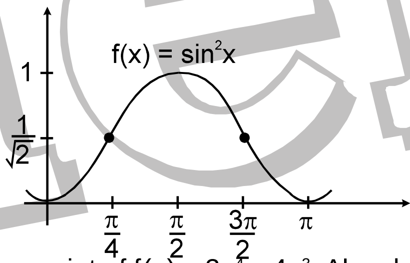


Example : Find the points of inflection of the function $f(x) = \sin^2x \quad x \in [0, 2\pi]$

Solution. $f(x) = \sin^2x$
$f'(x) = \sin2x$
$f''(x) = 2\cos2x$

$f''(0) = 0 \quad \Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}$

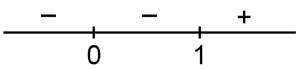
both these points are inflection points as sing of $f''(x)$ change but $f'(x)$ does not changes about these points.



Example : Find the inflection point of $f(x) = 3x^4 - 4x^3$. Also draw the graph of f(x) giving due importance to maxima, minima and concavity.

Solution. $f(x) = 3x^4 - 4x^3$
$f'(x) = 12x^3 - 12x^2$
$f'(x) = 12x^2(x - 1)$
$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1$

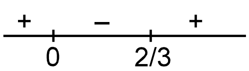
examining sign change of $f'(x)$



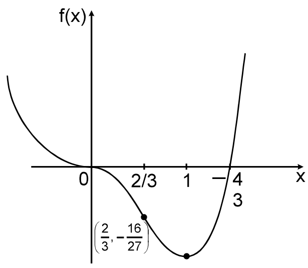
thus x = 1 is a point of local minima
$f''(x) = 12(3x^2 - 2x)$
$f''(x) = 12x(3x - 2)$

$f''(x) = 0 \quad \Rightarrow \quad x = 0, \frac{2}{3}$.

Again examining sign of $f''(x)$

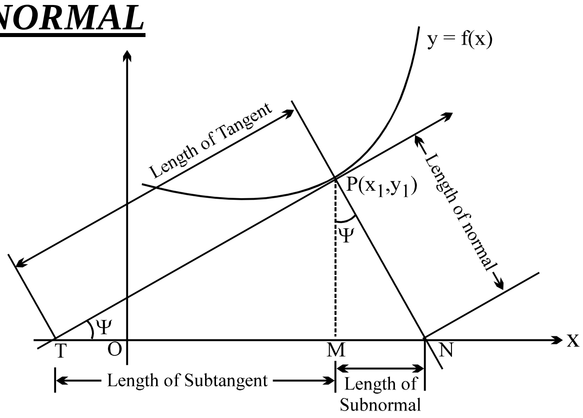


thus $x = 0, \frac{2}{3}$ are the inflection points
Hence the graph of f(x) is



SHORT REVISION

TANGENT & NORMAL



THINGS TO REMEMBER :

I The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically

$f'(x_1) = \left.\dfrac{dy}{dx}\right]_{x_1 y_1}$ = Slope of tangent at $P(x_1\,y_1) = m$ (say).

II Equation of tangent at (x_1, y_1) is ;

$$y - y_1 = \left.\frac{dy}{dx}\right]_{x_1 y_1}(x - x_1).$$

III Equation of normal at (x_1, y_1) is ;

$$y - y_1 = -\frac{1}{\left.\dfrac{dy}{dx}\right]_{x_1 y_1}}(x - x_1).$$

NOTE :

1. The point $P(x_1, y_1)$ will satisfy the equation of the curve & the equation of tangent & normal line.
2. If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.
3. If the tangent at any point on the curve is parallel to the axis of y, then dy/dx = ∞ or dx/dy = 0.
4. If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ± 1.
5. If the tangent at any point makes equal intercept on the coordinate axes then dy/dx = – 1.
6. Tangent to a curve at the point $P(x_1, y_1)$ can be drawn even through dy/dx at P does not exist. e.g. x = 0 is a tangent to $y = x^{2/3}$ at (0, 0).
7. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. x + y = 0 and x – y = 0.

IV Angle of intersection between two curves is defined as the angle between the 2 tangents drawn to the 2 curves at their point of intersection. If the angle between two curves is 90° every where then they are called **ORTHOGONAL** curves.

V **(a)** Length of the tangent (PT) = $\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$ **(b)** Length of Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+[f'(x_1)]^2}$ **(d)** Length of Subnormal (MN) = $y_1 f'(x_1)$

VI **DIFFERENTIALS :**
The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, y = tan x then $dy = \sec^2 x\, dx$.
In general $dy = f'(x)\,dx$.
Note that : d (c) = 0 where 'c' is a constant.
d (u + v – w) = du + dv – dw d (u v) = u d v + v d u

Note :1. For the independent variable 'x', increment Δ x and differential d x are equal but this is not the case with the dependent variable 'y' i.e. Δ y ≠ d y.

2. The relation $dy = f'(x)\,dx$ can be written as $\dfrac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

EXERCISE–1

Q.1 Find the equations of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2).

Q.2 Find the point of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is intersected by the curve xy = 1 – y.

Q.3 Find all the lines that pass through the point (1, 1) and are tangent to the curve represented parametrically as $x = 2t - t^2$ and $y = t + t^2$.

Q.4 In the curve $x^a y^b = K^{a+b}$, prove that the portion of the tangent intercepted between the coordinate axes is divided at its point of contact into segments which are in a constant ratio. (All the constants being positive).

Q.5 A straight line is drawn through the origin and parallel to the tangent to a curve $\dfrac{x+\sqrt{a^2-y^2}}{a} = \ln\left(\dfrac{a+\sqrt{a^2-y^2}}{y}\right)$ at an arbitary point M. Show that the locus of the point P of intersection of the straight line through the origin & the straight line parallel to the x-axis & passing through the point M is $x^2 + y^2 = a^2$.

Q.6 Prove that the segment of the tangent to the curve $y = \dfrac{a}{2}\ln\dfrac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}} - \sqrt{a^2-x^2}$ contained between

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the y-axis & the point of tangency has a constant length.

Q.7 A function is defined parametrically by the equations

$$f(t) = x = \begin{cases} 2t + t^2 \sin\frac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \quad \text{and } g(t) = y = \begin{cases} \frac{1}{t}\sin t^2 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Find the equation of the tangent and normal at the point for t = 0 if exist.

Q.8 Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$.

Q.9 (a) Find the value of n so that the subnormal at any point on the curve $xy^n = a^{n+1}$ may be constant.

(b) Show that in the curve $y = a.\ ln(x^2 - a^2)$, sum of the length of tangent & subtangent varies as the product of the coordinates of the point of contact.

Q.10 Prove that the segment of the normal to the curve $x = 2a \sin t + a \sin t \cos^2 t$; $y = -a \cos^3 t$ contained between the co-ordinate axes is equal to 2a.

Q.11 Show that the normals to the curve $x = a(\cos t + t \sin t)$; $y = a(\sin t - t \cos t)$ are tangent lines to the circle $x^2 + y^2 = a^2$.

Q.12 The chord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point x = 2 and is bisected by that point. Find 'a'.

Q.13 If the tangent at the point (x_1, y_1) to the curve $x^3 + y^3 = a^3$ $(a \neq 0)$ meets the curve again in (x_2, y_2) then show that $\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$.

Q.14 Determine a differentiable function $y = f(x)$ which satisfies $f'(x) = [f(x)]^2$ and $f(0) = -\frac{1}{2}$. Find also the equation of the tangent at the point where the curve crosses the y-axis.

Q.15 If p_1 & p_2 be the lengths of the perpendiculars from the origin on the tangent & normal respectively at any point (x, y) on a curve, then show that $\left.\begin{matrix} p_1 = |x \sin\Psi - y \cos\Psi| \\ p_2 = |x \cos\Psi + y \sin\Psi| \end{matrix}\right]$ where $\tan\Psi = \frac{dy}{dx}$. If in the above case, the curve be $x^{2/3} + y^{2/3} = a^{2/3}$ then show that : $4p_1^2 + p_2^2 = a^2$.

Q.16 The curve $y = ax^3 + bx^2 + cx + 5$, touches the x-axis at $P(-2, 0)$ & cuts the y-axis at a point Q where its gradient is 3. Find a, b, c.

Q.17 The tangent at a variable point P of the curve $y = x^2 - x^3$ meets it again at Q. Show that the locus of the middle point of PQ is $y = 1 - 9x + 28x^2 - 28x^3$.

Q.18 Show that the distance from the origin of the normal at any point of the curve $x = a e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)$ & $y = a e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$ is twice the distance of the tangent at the point from the origin.

Q.19 Show that the condition that the curves $x^{2/3} + y^{2/3} = c^{2/3}$ & $\mathbf{(x^2/a^2) + (y^2/b^2) = 1}$ may touch if c = a + b.

Q.20 The graph of a certain function *f* contains the point (0, 2) and has the property that for each number '*p*' the line tangent to $y = f(x)$ at $(p, f(p))$ intersect the x-axis at p + 2. Find $f(x)$.

Q.21 A curve is given by the equations $x = at^2$ & $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P & Q. Show that the locus of the point of intersection of the tangents at P & Q is $4y^2 = 3ax - a^2$.

Q.22(a) Show that the curves $\frac{x^2}{a^2 + K_1} + \frac{y^2}{b^2 + K_1} = 1$ & $\frac{x^2}{a^2 + K_2} + \frac{y^2}{b^2 + K_2} = 1$ intersect orthogonally.

(b) Find the condition that the curves $\frac{x^2}{a} + \frac{y^2}{b} = 1$ & $\frac{x^2}{a'} + \frac{y^2}{b'} = 1$ may cut orthogonally.

Q.23 Show that the angle between the tangent at any point 'A' of the curve $ln(x^2 + y^2) = C \tan^{-1}\frac{y}{x}$ and the line joining A to the origin is independent of the position of A on the curve.

Q.24 For the curve $x^{2/3} + y^{2/3} = a^{2/3}$, show that $|z|^2 + 3p^2 = a^2$ where $z = x + iy$ & p is the length of the perpendicular from (0, 0) to the tangent at (x, y) on the curve.

Q.25 A and B are points of the parabola $y = x^2$. The tangents at A and B meet at C. The median of the triangle ABC from C has length 'm' units. Find the area of the triangle in terms of 'm'.

EXERCISE–2

RATE MEASURE AND APPROXIMATIONS

Q.1 Water is being poured on to a cylindrical vessel at the rate of 1 m^3/min. If the vessel has a circular base of radius 3 m, find the rate at which the level of water is rising in the vessel.

Q.2 A man 1.5 m tall walks away from a lamp post 4.5 m high at the rate of 4 km/hr.

(i) how fast is the farther end of the shadow moving on the pavement ?

(ii) how fast is his shadow lengthening ?

Q.3 A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate.

Q.4 An inverted cone has a depth of 10 cm & a base of radius 5 cm. Water is poured into it at the rate of 1.5 cm^3/min. Find the rate at which level of water in the cone is rising, when the depth of water is 4cm.

Q.5 A water tank has the shape of a right circular cone with its vertex down. Its altitude is 10 cm and the radius of the base is 15 cm. Water leaks out of the bottom at a constant rate of 1cu. cm/sec. Water is poured into the tank at a constant rate of C cu. cm/sec. Compute C so that the water level will be rising at the rate of 4 cm/sec at the instant when the water is 2 cm deep.

Q.6 Sand is pouring from a pipe at the rate of 12 cc/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6th of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm.

Q.7 An open Can of oil is accidently dropped into a lake ; assume the oil spreads over the surface as a circular disc of uniform thickness whose radius increases steadily at the rate of 10 cm/sec. At the moment when the radius is 1 meter, the thickness of the oil slick is decreasing at the rate of 4 mm/sec, how fast is it decreasing when the radius is 2 meters.

Q.8 Water is dripping out from a conical funnel of semi vertical angle $\pi/4$, at the uniform rate of 2 cm^3/sec through a tiny hole at the vertex at the bottom. When the slant height of the water is 4cm, find the rate of decrease of the slant height of the water.

Q.9 An air force plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is R Km, how fast the area of the earth, visible from the plane increasing at 3min after it started ascending. Take visible area A $= \frac{2\pi R^2 h}{R+h}$ Where h is the height of the plane in kms above the earth.

Q.10 A variable Δ ABC in the xy plane has its orthocentre at vertex 'B' , a fixed vertex 'A' at the origin and the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when $t = \frac{7}{2}$ sec.

Q.11 A circular ink blot grows at the rate of 2 cm^2 per second. Find the rate at which the radius is increasing after $2\frac{6}{11}$ seconds. Use $\pi = \frac{22}{7}$.

Q.12 Water is flowing out at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius R = 13 m. The volume of water in the hemispherical bowl is given by $V = \frac{\pi}{3} \cdot y^2(3R - y)$ when the water is y meter deep. Find

(a) At what rate is the water level changing when the water is 8 m deep.

(b) At what rate is the radius of the water surface changing when the water is 8 m deep.

Q.13 If in a triangle ABC, the side 'c' and the angle 'C' remain constant, while the remaining elements are changed slightly, show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$.

Q.14 At time t > 0, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At t = 0, the radius of the sphere is 1 unit and at t = 15 the radius is 2 units.

(a) Find the radius of the sphere as a function of time t.

(b) At what time t will the volume of the sphere be 27 times its volume at t = 0.

Q.15 Use differentials to a approximate the values of ; (a) $\sqrt{25.2}$ and (b) $\sqrt[3]{26}$.

EXERCISE–3

Q.1 Find the acute angles between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their point of intersection.

Q.2 Find the equation of the straight line which is tangent at one point and normal at another point of the curve, $x = 3t^2$, $y = 2t^3$. [REE 2000 (Mains) 5 out of 100]

Q.3 If the normal to the curve, y = f(x) at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x–axis. Then $f'(3) =$

(A) – 1 (B) $-\frac{3}{4}$ (C) $\frac{4}{3}$ (D) 1

Q.4 The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is(are) [JEE 2002 (Scr.), 3]

(A) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (B) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$ (C) (0, 0) (D) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

Q.5 Tangent to the curve $y = x^2 + 6$ at a point P (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are [JEE 2005 (Scr.), 3]

(A) (– 6, –11) (B) (–9, –13) (C) (– 10, – 15) (D) (–6, –7)

EXERCISE–4

PART - (A) Only one correct option

1. Water is poured into an inverted conical vessel of which the radius of the base is 2 m and height 4 m, at the rate of 77 litre/minute. The rate at which the water level is rising at the instant when the depth is 70 cm is : (use π = 22/7)

(A) 10 cm/min (B) 20 cm/min (C) 40 cm/min (D) none

2. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is
(A) $3\sqrt{3}$ sq. units (B) $2\sqrt{3}$ sq. units (C) $4\sqrt{3}$ sq. units (D) $\sqrt{3}$ sq. units

3. The line which is parallel to x-axis and crosses the curve $y = \sqrt{x}$ at an angle of $\frac{\pi}{4}$ is
(A) y = – 1/2 (B) x = 1/2 (C) y = 1/4 (D) y = 1/2

4. If at any point on a curve the subtangent and subnormal are equal, then the tangent is equal to
(A) oridinate (B) $\sqrt{2}$ ordinate (C) $\sqrt{2\,(\text{ordinate})}$ (D) none of these

5. If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then the value of a is
(A) 1/2 (B) 1/3 (C) 2 (D) 3

6. For a curve $\frac{(\text{length of normal})^2}{(\text{length of tangent})^2}$ is equal to
(A) (subnormal) / (subtangent) (B) (subtangent) / (subnormal)
(C) subnormal/(subtangent)^2 (D) none of these

7. If the tangent at each point of the curve $y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$ makes an acute angle with the positive direction of x-axis, then
(A) $a \ge 1$ (B) $-1 \le a \le 1$ (C) $a \le -1$ (D) none of these

8. Equation of normal drawn to the graph of the function defined as $f(x) = \frac{\sin x^2}{x}$, $x \ne 0$ and $f(0) = 0$ at the origin is:
(A) x + y = 0 (B) x – y = 0 (C) y = 0 (D) x = 0

9. All points on the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ at which the tangents are parallel to the axis of x, lie on a
(A) circle (B) parabola (C) line (D) none of these

10. The point(s) of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is intersected by the curve $xy = 1 - y$ is/are given by:
(A) (0, – 1) (B) (0, 1) (C) (1, 1) (D) none of these

11. The ordinate of $y = (a/2)\,(e^{x/a} + e^{-x/a})$ is the geometric mean of the length of the normal and the quantity:
(A) a/2 (B) a (C) e (D) none of these

12. The curves $x^3 + p\,xy^2 = -2$ and $3x^2y - y^3 = 2$ are orthogonal for:
(A) p = 3 (B) p = – 3 (C) no value of p (D) p = ± 3

13. If the area of the triangle included between the axes and any tangent to the curve $x^n y = a^n$ is constant, then n is equal to
(A) 1 (B) 2 (C) 3/2 (D) 1/2

14. A curve with equation of the form $y = ax^4 + bx^3 + cx + d$ has zero gradient at the point (0, 1) and also touches the x – axis at the point (– 1, 0) then the values of x for which the curve has a negative gradient are:
(A) x > – 1 (B) x < 1 (C) x < – 1 (D) $-1 \le x \le 1$

15. If the tangent at P of the curve $y^2 = x^3$ intersects the curve again at Q and the straight lines OP, OQ make angles α, β with the x-axis, where 'O' is the origin, then $\tan\alpha/\tan\beta$ has the value equal to:
(A) – 1 (B) – 2 (C) 2 (D) $\sqrt{2}$

PART - (B) One or more than one correct options

16. Consider the curve $f(x) = x^{1/3}$, then
(A) the equation of tangent at (0, 0) is x = 0 (B) the equation of normal at (0, 0) is y = 0
(C) normal to the curve does not exist at (0, 0) (D) f(x) and its inverse meet at exactly 3 points.

17. The equation of normal to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ $(n \in N)$ at the point with abscissa equal to 'a' can be:
(A) $ax + by = a^2 - b^2$ (B) $ax + by = a^2 + b^2$
(C) $ax - by = a^2 - b^2$ (D) $bx - ay = a^2 - b^2$

18. If the line, ax + by + c = 0 is a normal to the curve xy = 2, then:
(A) a < 0, b > 0 (B) a > 0, b < 0 (C) a > 0, b > 0 (D) a < 0, b < 0

19. In the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$, at point (2, – 1)
(A) length of subtangent is 7/6. (B) slope of tangent = 6/7
(C) length of tangent = $\sqrt{(85)}/6$ (D) none of these

20. If y = f(x) be the equation of a parabola which is touched by the line y = x at the point where x = 1. Then
(A) $f'(1) = 1$ (B) $f'(0) = f'(1)$ (C) $2f(0) = 1 - f'(0)$ (D) $f(0) + f'(0) + f''(0) = 1$

21. If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts off intercepts α, β on co-ordinate axes, where $\alpha^2 + \beta^2 = 61$, then the value of 'a' is equal to:
(A) 20 (B) 25 (C) 30 (D) – 30

22. The curves $ax^2 + by^2 = 1$ and $Ax^2 + By^2 = 1$ intersect orthogonally, then
(A) $\frac{1}{a} + \frac{1}{A} = \frac{1}{b} + \frac{1}{B}$ (B) $\frac{1}{a} - \frac{1}{A} = \frac{1}{b} - \frac{1}{B}$ (C) $\frac{1}{a} + \frac{1}{b} = \frac{1}{B} - \frac{1}{A}$ (D) $\frac{1}{a} - \frac{1}{b} = \frac{1}{A} - \frac{1}{B}$

EXERCISE–5

1. Find the parameters a, b, c if the curve $y = a\,x^2 + b\,x + c$ is to pass through the point (1, 2) and is to be tangent to the line y = x at the origin.

2. If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then find coordinates of P

3. If the relation between subnormal SN and subtangent ST at any point S on the curve $by^2 = (x + a)^3$ is $p(SN) = q\,(ST)^2$, then find value of $\frac{p}{q}$ in terms of b and a.

4. In the curve $x = a\left(\cos t + \log \tan \frac{1}{2} t\right)$, $y = a \sin t$, show that the portion of the tangent between the point of contact and the x – axis is of constant length.

5. Find the angle of intersection of the following curves:
(i) $2y^2 = x^3$ & $y^2 = 32x$ (ii) $y = 2\sin^2 x$ and $y = \cos 2x$ at $x = \pi/6$
(iii) $y = 4 - x^2$ & $y = x^2$

6. The length x of rectangle is decreasing at a rate of 3 cm/min and the width y is increasing at the rate of 2 cm/min. when x = 10 cm and y = 6 cm, find the rates of changes of (i) the perimeter, and (ii) the area of the ractangle.

7. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate.

8. Prove that the straight line, $x \cos \alpha + y \sin \alpha = p$ will be a tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$.

9. Show that the normal to any point of the curve $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ is at a constant distance from the origin.

10. Show that the condition, that the curves $x^{2/3} + y^{2/3} = c^{2/3}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may touch, if $c = a + b$.

11. Find the equation of axes of the conic $5x^2 + 4xy + 2y^2 = 1$.

12. Find the abscissa of the point on the curve, $xy = (c + x)^2$ the normal at which cuts off numerically equal intercepts from the axes of co-ordinates.

13. In the curve $x^a y^b = K^{a+b,}$ prove that the portion of the tangent intercepted between the coordinate axes is divided at its point of contact into segments which are in a constant ratio. (All the constants being positive).

14. The tangent to curve $y = x - x^3$ at point P meets the curve again at Q. Prove that one point of trisection of PQ lies on y-axis. Find locus of other point of trisection

15. A straight line is drawn through the origin and parallel to the tangent to a curve $\frac{x + \sqrt{a^2 - y^2}}{a} = \ell n\left(\frac{a + \sqrt{a^2 - y^2}}{y}\right)$ at an arbitary point M. Show that the locus of the point P of intersection of the straight line & the straight line parallel to the x-axis & passing through the point M is $x^2 + y^2 = a^2$.

16. Find the possible values of a such that the inequality $3 - x^2 > |x - a|$ has atleast one negative solution.

17. Consider the family of circles $x^2 + y^2 = r^2$, $2 < r < 5$. In the first quadrant, the common tangents to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the co-ordinate axes at A and B, then find the equation of the locus of the mid-point of AB. **[IIT – 1999]**

18. Let T_1, T_2 be two tangents drawn from (–2, 0) onto the circle C : $x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further; find the equations of all possible common tangents to these circles, when taken two at a time. **[IIT – 1999]**

19. An inverted cone of height H and radius R is pointed at bottom. It is filled with a volatile liquid completely. If the rate of evaporation is directly proportional to the surface area of the liquid in contact with air (constant of proportionality k > 0). Find the time in which whole liquid evaporates. **[IIT – 2003, 4]**

20. If $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tengent to the curve $y = f(x)$ at the point (1, 2). **[IIT – 2005, 2]**

MONOTONOCITY

(Significance of the sign of the first order derivative)

DEFINITIONS :

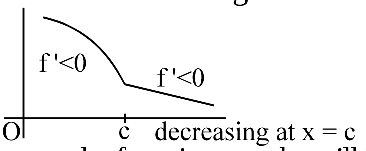
1. A function f(x) is called an Increasing Function at a point x = a if in a sufficiently small neighbourhood around x = a we have $\left.\begin{matrix} f(a+h) > f(a) \text{ and} \\ f(a-h) < f(a) \end{matrix}\right\}$ increasing;

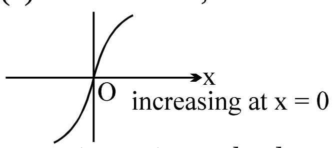
Similarly decreasing if $\left.\begin{matrix} f(a+h) < f(a) \text{ and} \\ f(a-h) > f(a) \end{matrix}\right\}$ decreasing.

2. A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.

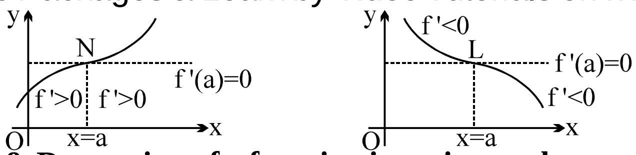
3. A function which in a given interval is increasing or decreasing is called **"Monotonic"** in that interval.

4. **Tests for increasing and decreasing of a function at a point :**
If the derivative $f'(x)$ is positive at a point x = a, then the function f (x) at this point is increasing. If it is negative, then the function is decreasing. Even if f'(a) is not defined, f can still be increasing or decreasing.





Note : If $f'(a) = 0$, then for x = a the function may be still increasing or it may be decreasing as shown. It has to be identified by a seperate rule. e.g. $f(x) = x^3$ is increasing at every point.
Note that, $\mathbf{dy/dx = 3x^2}$.



5. Tests for Increasing & Decreasing of a function in an interval :

Sufficiency Test : If the derivative function $f'(x)$ in an interval (a, b) is every where positive, then the function $f(x)$ in this interval is Increasing ;
If $f'(x)$ is every where negative, then $f(x)$ is Decreasing.

General Note :

(1) If a function is invertible it has to be either increasing or decreasing.

(2) If a function is continuous the intervals in which it rises and falls may be separated by points at which its derivative fails to exist.

(3) If f is increasing in [a, b] and is continuous then f (b) is the greatest and f (c) is the least value of f in [a, b]. Similarly if f is decreasing in [a, b] then f (a) is the greatest value and f (b) is the least value.

6.

(a) ROLLE'S THEOREM :

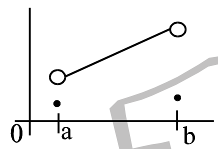
Let f(x) be a function of x subject to the following conditions :

(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$.

Then there exists at least one point x = c such that $a < c < b$ where $f'(c) = 0$.
Note that if f is not continuous in closed [a, b] then it may lead to the adjacent graph where all the 3 conditions of Rolles will be valid but the assertion will not be true in (a, b).



(b) LMVT THEOREM :

Let f(x) be a function of x subject to the following conditions :

(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) \ne f(b)$.

Then there exists at least one point x = c such that $a < c < b$ where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrically, the slope of the secant line joining the curve at x = a & x = b is equal to the slope of the tangent line drawn to the curve at x = c. Note the following :

✍ Rolles theorem is a special case of LMVT since
$f(a) = f(b) \Rightarrow f'(c) = \dfrac{f(b) - f(a)}{b - a} = 0.$

Note : Now [f (b) – f (a)] is the change in the function f as x changes from a to b so that [f (b) – f (a)] / (b – a) is the *average rate of change* of the function over the interval [a, b]. Also f '(c) is the actual rate of change of the function for x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.
This interpretation of the theorem justifies the name "Mean Value" for the theorem.

(c) Application Of Rolles Theorem For Isolating The Real Roots Of An Equation f (x)=0

Suppose a & b are two real numbers such that ;

(i) f(x) & its first derivative $f'(x)$ are continuous for $a \le x \le b$.

(ii) f(a) & f(b) have opposite signs.

(iii) $f'(x)$ is different from zero for all values of x between a & b.

Then there is one & only one real root of the equation f(x) = 0 between a & b.

EXERCISE–6

Q.1 Find the intervals of monotonocity for the following functions & represent your solution set on the number line.
(a) $f(x) = 2.\ e^{x^2-4x}$ (b) $f(x) = e^x/x$ (c) $f(x) = x^2 e^{-x}$ (d) $f(x) = 2x^2 - \ln|x|$
Also plot the graphs in each case.

Q.2 Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$

Q.3 Find the intervals of monotonocity of the function
(a) $f(x) = \sin x - \cos x$ in $x \in [0, 2\pi]$ (b) $g(x) = 2\sin x + \cos 2x$ in $(0 \le x \le 2\pi)$.

Q.4 Show that, $x^3 - 3x^2 - 9x + 20$ is positive for all values of $x > 4$.

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Q.5 Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \begin{cases} \max\{f(t) : 0 \le t \le x\} & , 0 \le x \le 1 \\ 3 - x & , 1 < x \le 2 \end{cases}$

Discuss the conti. & differentiability of g(x) in the interval (0,2).

Q.6 Find the set of all values of the parameter 'a' for which the function, $f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in R$ and has no critical points for all $x \in R$.

Q.7 Find the greatest & the least values of the following functions in the given interval if they exist.

(a) $f(x) = \sin^{-1}\dfrac{x}{\sqrt{x^2+1}} - \ln x$ in $\left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$ (b) $y = x^x$ in $(0, \infty)$ (c) $y = x^5 - 5x^4 + 5x^3 + 1$ in $[-1, 2]$

Q.8 Find the values of 'a' for which the function $f(x) = \sin x - a\sin 2x - \dfrac{1}{3}\sin 3x + 2ax$ increases throughout the number line.

Q.9 Prove that $f(x) = \displaystyle\int_2^{e^x}\left(9\cos^2(2\ln t) - 25\cos(2\ln t) + 17\right)dt$ is always an increasing function of x, $\forall\, x \in R$

Q.10 If $f(x) = \left(\dfrac{a^2-1}{3}\right)x^3 + (a-1)x^2 + 2x + 1$ is monotonic increasing for every $x \in R$ then find the range of values of 'a'.

Q.11 Find the set of values of 'a' for which the function,

$f(x) = \left(1 - \dfrac{\sqrt{21 - 4a - a^2}}{a + 1}\right)x^3 + 5x + \sqrt{7}$ is increasing at every point of its domain.

Q.12 Find the intervals in which the function $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3,\ 0 \le x \le \pi$; is monotonically increasing or decreasing.

Q.13 Find the range of values of 'a' for which the function $f(x) = x^3 + (2a + 3)x^2 + 3(2a + 1)x + 5$ is monotonic in R. Hence find the set of values of 'a' for which f (x) in invertible.

Q.14 Find the value of $x > 1$ for which the function

$F(x) = \displaystyle\int_x^{x^2}\frac{1}{t}\ln\left(\frac{t-1}{32}\right)dt$ is increasing and decreasing.

Q.15 Find all the values of the parameter 'a' for which the function ; $f(x) = 8ax - a\sin 6x - 7x - \sin 5x$ increases & has no critical points for all $x \in R$.

Q.16 If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ monotonically increases for every $x \in R$ then find the range of values of 'a'.

Q.17 Construct the graph of the function $f(x) = -\left|\dfrac{x^2-9}{x+3} - x + \dfrac{2}{x-1}\right|$ and comment upon the following

(a) Range of the function,
(b) Intervals of monotonocity,
(c) Point(s) where f is continuous but not diffrentiable,
(d) Point(s) where f fails to be continuous and nature of discontinuity.
(e) Gradient of the curve where f crosses the axis of y.

Q.18 Prove that, $x^2 - 1 > 2x\ln x > 4(x - 1) - 2\ln x$ for $x > 1$.

Q.19 Prove that $\tan^2 x + 6\ln\sec x + 2\cos x + 4 > 6\sec x$ for $x \in \left(\dfrac{3\pi}{2}, 2\pi\right)$.

Q.20 If $ax^2 + (b/x) \ge c$ for all positive x where $a > 0$ & $b > 0$ then show that $27ab^2 \ge 4c^3$.

Q.21 If $0 < x < 1$ prove that $y = x\ln x - (x^2/2) + (1/2)$ is a function such that $d^2y/dx^2 > 0$. Deduce that $x\ln x > (x^2/2) - (1/2)$.

Q.22 Prove that $0 < x.\sin x - (1/2)\sin^2 x < (1/2)(\pi - 1)$ for $0 < x < \pi/2$.

Q.23 Show that $x^2 > (1 + x)[\ln(1 + x)]^2\ \forall\, x > 0$.

Q.24 Find the set of values of x for which the inequality $\ln(1 + x) > x/(1 + x)$ is valid.

Q.25 If $b > a$, find the minimum value of $\left|(x-a)^3\right| + \left|(x-b)^3\right|$, $x \in R$.

EXERCISE–7

Q.1 Verify Rolles throrem for $f(x) = (x - a)^m(x - b)^n$ on [a, b] ; m, n being positive integer.

Q.2 Let $f : [a, b] \to R$ be continuous on [a, b] and differentiable on (a, b). If f (a) < f (b), then show that $f'(c) > 0$ for some $c \in (a, b)$.

Q.3 Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, 0< c <1 such that f(c) = 0.